

Coherently enhanced measurements in classical mechanics

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We show that the recently discovered quantum-enhanced measurement protocol of coherent averaging [1] that is capable of achieving Heisenberg-limited sensitivity without using entanglement, has a classical analogue. The classical protocol uses N harmonic oscillators coupled to a central oscillator and one measures the signal from the latter. We propose an application to the measurement of very weak interactions, and, in particular, a novel route to measuring the gravitational constant with enhanced precision.

A common practice for increasing the signal to noise ratio in a measurement is to measure identically prepared systems N times and average the measurement results. This typically leads to a scaling of the sensitivity (i.e. the smallest resolvable change in a parameter that we want to measure) as $1/\sqrt{N}$, a scaling that is known in quantum measurement theory as the “shot noise limit” or the “standard quantum limit” (SQL), even though there is nothing genuinely quantum about this scaling: It holds whenever the central limit theorem applies.

With the rise of quantum information theory, the exciting possibility that the $1/\sqrt{N}$ behavior might be improved upon has received large attention [2–23]. It was shown that if one puts the N systems into an entangled state, a scaling as $1/N$ can be achieved, known as the “Heisenberg limit” (HL) [3]. Examples include the use of NOON states in a Mach-Zehnder interferometer [24, 25], or squeezed spin states for magnetometers based on atomic vapors [26]. Unfortunately, the entangled states required in these schemes are very unstable and prone to decoherence. Experiments with NOON states showing a slight improvement over the SQL have not surpassed yet the stage of more than a few entangled photons [2, 27]. In fact, the situation here is much more unfavorable than even for a quantum computer: whereas for the latter it should be enough to fully control a few hundred to a few thousand qubits in order to outperform existing classical computers for specific tasks such as factoring [28] or data fitting [29], classical experiments such as LIGO have already sensitivities of the order $10^{-22}/\sqrt{\text{Hz}}$ [30, 31]. To compete with such performance using an entangled state of N particles, one would have to entangle a macroscopic number of particles (or create a NOON state with a macroscopic number of photons), which seems out of reach considering the experimental difficulties of creating a NOON state with just 4 photons. Also, from theoretical grounds, it has become clear that for NOON states the slightest amount of decoherence leads back to the SQL scaling for sufficiently large N [32–34]. For niche applications, such in biological systems that require low intensities, these methods may nevertheless be interesting [20].

The commonly held belief that entanglement is necessary for reaching the HL is based on propagation of the quantum mechanical state of N distinguishable particles with a very simple hamiltonian: $H = \sum_{i=1}^N h_i(x)$, where $h_i(x)$ is a single particle hamiltonian [3]. No interactions between different particles are considered. Recently it was shown that a system with k -body interactions offers a scaling of the sensitivity as $\Delta_{\Psi}^2 J \propto 1/N^{k-1/2}$ without initial entanglement (and $1/N^k$ with initial entanglement [4–11]), even though interactions themselves will ultimately have to scale down with N if the total energy is to remain an extensive quantity. The k -body interactions typically lead to squeezed states and resemble in this respect the earliest examples of quantum-enhanced measurements that proposed the use of squeezed light [35, 36]. With indistinguishable particles, as obtained naturally e.g. from a Bose-Einstein condensate, one may avoid entangling the particles as well for surpassing the SQL limit, even though the definition of entanglement is more tricky in this case [37].

Another possibility is to have N distinguishable systems interact with a $N+1$ st system and read out the latter [38, 39]. This method has the advantage that the total system needs to accommodate only N interaction terms, such that at least in principle the interaction itself may be independent of N . Furthermore, the scaling with N appears to be stable under local decoherence, and even decoherence itself can be used as a signal, if the $N+1$ st system is an environment. The effect can be understood as “coherent averaging”: a phase accumulates in the state of the $N+1$ st system from the interaction with the N other systems. No entanglement is needed. These properties make one wonder whether a classical analogue of this mechanism exists. In the present Letter we show that this is indeed true. More specifically, we show that there is a phase accumulation mechanism in the classical motion of a central harmonic oscillator interacting with N other harmonic oscillators that is completely analogous to the quantum mechanical scenario. The phase accumulation allows one to achieve a sensitivity that scales as $1/N$, just as in the quantum

case with Heisenberg-limited sensitivity, even though there is of course nothing quantum. Thus, while the found sensitivity is, contrary to the quantum case, not the expression of a generalized Heisenberg uncertainty relation, it shows nevertheless that even in the classical realm there are situations where one can improve upon the venerated averaging of N independent measurement results by coupling the same resources “coherently” to a $N + 1$ st system, and measuring the latter.

Model. Consider a classical harmonic oscillator with frequency $\omega_0 \equiv \Omega$ harmonically coupled to N other harmonic oscillators with frequency ω_i , $i = 1, \dots, N$. The Hamilton function (with masses $m_i = 1$) reads

$$H = \frac{1}{2} \sum_{i=0}^N (p_i^2 + \omega_i^2 q_i^2) + \frac{1}{2} \xi^2 \sum_{i=1}^N (q_i - q_0)^2, \quad (1)$$

where p_i and q_i are the canonical momenta and coordinates, respectively, and ξ^2 denotes the coupling strength, such that ξ has the dimension of a frequency. This is the parameter we want to determine. The oscillators need not be mechanical oscillators, of course.

The total potential energy in the problem can be rewritten as a quadratic form,

$$V(\mathbf{q}) \equiv H - \sum_{i=0}^N p_i^2/2 = \frac{1}{2} \mathbf{q}^t \mathbf{C} \mathbf{q}, \quad (2)$$

with $\mathbf{q}^t = (q_0, \dots, q_N)$, and

$$\mathbf{C} = \begin{pmatrix} \Omega^2 + N\xi^2 & -\xi^2 & \dots & -\xi^2 \\ -\xi^2 & \omega_1^2 + \xi^2 & 0 \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ -\xi^2 & 0 & \dots & \omega_N^2 + \xi^2 \end{pmatrix}. \quad (3)$$

The problem can be solved by diagonalizing \mathbf{C} . We are interested in very small couplings, $\xi^2 \ll \omega_i^2$, $i = 0, \dots, N$, where diagonalization can be performed perturbatively, starting from the uncoupled (squared) eigenfrequencies $\lambda_i = \omega_i^2$, $i = 0, \dots, N$. Furthermore, we will restrict ourselves to the situation where the ω_i are narrowly distributed about a central frequency $\bar{\omega}$, and sufficiently off-resonant from Ω , such that non-degenerate perturbation theory (PT) will suffice to obtain the correction to Ω . To order $\mathcal{O}(\xi^2)$ we have $\lambda_0 = \Omega^2 + N\xi^2 + \mathcal{O}(\xi^4)$ and $\lambda_l = \omega_l^2 + \xi^2 + \mathcal{O}(\xi^4)$, $l = 1, \dots, N$. The perturbed eigenmodes \mathbf{u}_l are summarized in the orthogonal transformation matrix $\mathbf{U} = (\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_N)$ to order $\mathcal{O}(\xi^2)$ as

$$\mathbf{U} = \begin{pmatrix} 1 & -\frac{\xi^2}{\omega_1^2 - \Omega^2} & \dots & -\frac{\xi^2}{\omega_N^2 - \Omega^2} \\ \frac{\xi^2}{\omega_1^2 - \Omega^2} & 1 & 0 & \dots \\ \vdots & 0 & \ddots & 0 \\ \frac{\xi^2}{\omega_N^2 - \Omega^2} & 0 & \dots & 1 \end{pmatrix}. \quad (4)$$

We consider two sources of uncertainty: *i.)* an uncertainty in the frequencies ω_i , $i = 1, \dots, N$, and *ii.)* time dependent noise.

i.) Uncertainty in frequencies. The equations of motion without external driving, $\ddot{q}_i + \sum_j C_{ij} q_j = 0$ are decoupled into $N + 1$ independent harmonic oscillators by the transformation $\mathbf{q} \rightarrow \tilde{\mathbf{q}} = \mathbf{U}^t \mathbf{q}$. Solving them and transforming back leads to the response of the central oscillator. In order to simplify expressions, we specialize to vanishing initial speeds for all oscillators, $\dot{q}_j(0) = 0$, $j = 0, \dots, N$. We will furthermore assume $N\xi^2 \ll \Omega^2$, such that $\sqrt{\lambda_0} = \Omega(1 + N\xi^2/(2\Omega^2)) + \mathcal{O}(\xi^4)$. This limits N , but for very small ξ^2 , N can become very large (see also the comments below for the validity beyond PT). To $\mathcal{O}(\xi^2)$ we then get

$$q_0(t) = q_0(0) \cos((\Omega + \frac{N\xi^2}{2\Omega})t) + \xi^2 \sum_{j=1}^N \frac{q_j(0)}{\omega_j^2 - \Omega^2} \times \left(\cos((\Omega + \frac{N\xi^2}{2\Omega})t) - \cos((\omega_j + \frac{\xi^2}{2\omega_j})t) \right). \quad (5)$$

The appearance of a phase shift that scales proportional to N and the parameter to be measured is reminiscent of phase superresolution [27]. This signal can be recovered by mixing the response of the central oscillator with a $\cos(\Omega t)$ signal corresponding to the unperturbed oscillator, i.e. one multiplies $q_0(t)$ with $\cos(\Omega t)$, which creates two signals, one that oscillates with the sum of the two frequencies, the other with the difference. The latter varies very slowly, and can be isolated with a low-pass filter. If we assume the spectral width of the low-pass filter to be much smaller than 2Ω and $\omega_j - \Omega$, the remaining signal $s(t)$ reads

$$s(t) = (q_0(0) + \xi^2 r(\{\omega_i\})) \cos\left(\frac{N\xi^2}{2\Omega}t\right),$$

$$\text{where } r(\{\omega_i\}) = \sum_{j=1}^N \frac{q_j(0)}{\omega_j^2 - \Omega^2} \quad (6)$$

is a random variable whose distribution is given by the distribution $P(\{\omega_j\})$ of the ω_j . The smallest uncertainty with which ξ^2 can be measured is given by [40]

$$\delta\xi_{min}^2 = \frac{\sigma(s(t))}{\sqrt{M} \left| \left\langle \frac{\partial s(t)}{\partial \xi^2} \right\rangle \right|}, \quad (7)$$

where $\langle \dots \rangle$ means average over $P(\{\omega_j\})$, $\sigma(s(t))$ is the standard deviation of $s(t)$ with respect to this distribution, and M is the number of measurements. It has the meaning of the smallest variation in ξ^2 that moves the average of the signal at least a distance given by the width of the distribution of the signal. A short calcula-

tion yields

$$\delta\xi_{min}^2 = \frac{\xi^2 \sigma(r) |\cos(\frac{N\xi^2}{2\Omega}t)|}{\sqrt{M} \left| \frac{N}{2\Omega}t (q_0(0) + \xi^2 \langle r \rangle) \sin(\frac{N\xi^2}{2\Omega}t) - \langle r \rangle \cos\left(\frac{N\xi^2}{2\Omega}t\right) \right|}. \quad (8)$$

If one waits long enough ($N\xi^2 t/(2\Omega) \gg 1$), the first term in the denominator dominates. If we set in addition $q_0(0) = 0$, we obtain the final result

$$\delta\xi_{min}^2 = \frac{1}{N} \frac{2\Omega}{\sqrt{M}t} \left| \cot\left(\frac{N\xi^2 t}{2\Omega}\right) \right| \frac{\sigma(r)}{\langle r \rangle}. \quad (9)$$

The prefactor $1/N$ identifies this minimal uncertainty under the noise process considered as “Heisenberg-limited”. Of course, this has nothing to do with Heisenberg’s uncertainty relation. Rather, we have considered a classical noise process (uncertainty in the original frequencies ω_i), but have found a way to reduce the resulting smallest uncertainty with which the parameter ξ^2 can be measured from a $1/\sqrt{N}$ scaling (that would be obtained by measuring it separately from each system i and the central oscillator, and then averaging) to a $1/N$ scaling. The process how this happens is completely analogous to the quantum mechanical collective phase accumulation described in [38, 39]: N systems interact with a common central system, and lead to an accumulated phase proportional to N and the parameter to be measured. This manifests itself in an oscillation with a frequency proportional to $N\xi^2$ in a corotating frame — the equivalent of homodyne detection in the quantum optical setting. It also leads to the same scaling with t , namely as $1/t$, and not the usual $1/\sqrt{t}$. It means that the sensitivity per square root of Hertz, $\delta\xi_{min}^2 \sqrt{t}$, still decreases as $1/\sqrt{t}$, just as in the Heisenberg limited quantum case. Different from the quantum-mechanical case is, however, the restriction to N such that $N\xi^2 \ll \Omega^2$. This is true beyond the validity of PT, as is seen by analysing the very strong coupling limit, where \mathbf{C} can again be diagonalized analytically. Nevertheless, the method proposed here may be advantageous for very small interactions, where N can be very large.

ii.) Time-dependent noise In the presence of time-dependent noise forces $f_i(t)$ acting on oscillator i , the equations of motion in the original oscillator coordinates q_i read

$$\ddot{q}_i + \sum_j C_{ij} q_j = f_i(t). \quad (10)$$

After transformation to the eigenmodes \tilde{q}_i defined through $q_j = \sum_l U_{jl} \tilde{q}_l$ we get

$$\ddot{\tilde{q}}_k + \lambda_k \tilde{q}_k = \sum_i U_{ki}^\dagger f_i(t) \equiv \tilde{f}_k(t). \quad (11)$$

A special solution of this equation can be found with the help of the Greens-function of the harmonic oscillator.

The back transformation to the original coordinates gives for the central oscillator

$$q_0(t) = q_0(0) \cos(\sqrt{\lambda_0}t) + \frac{\dot{q}_0(0)}{\sqrt{\lambda_0}} \sin(\sqrt{\lambda_0}t) + \int_0^t \frac{\sin \sqrt{\lambda_0}(t-t')}{\sqrt{\lambda_0}} f_0(t') dt' + \mathcal{O}(\xi^2). \quad (12)$$

The noise enters here already at order ξ^0 , i.e. perturbs even the uncoupled central oscillator. Nevertheless, we will now see that the phase accumulation of the central oscillator due to the coupling to the N other oscillators still leads to a $1/N$ scaling of the sensitivity.

We restrict ourselves again to $\dot{q}_j(0) = 0$ for all $j = 0, \dots, N$ and $N\xi^2 \ll \Omega^2$ such that $\sqrt{\lambda_0} = \Omega(1 + N\xi^2/(2\Omega^2)) + \mathcal{O}(\xi^4)$, and calculate the direct response to the noise. For simplicity we consider noise with zero average, $\langle f_0(t) \rangle = 0 \forall t$, where $\langle \dots \rangle$ means now average over the noise-process. We then have $\langle q_0(t) \rangle = q_0(0) \cos \sqrt{\lambda_0}t$ and $\sigma^2(q_0(t)) = \sigma^2(n(t))$, where

$$n(t) = \int_0^t \frac{\sin \sqrt{\lambda_0}(t-t')}{\sqrt{\lambda_0}} f_0(t') dt' \quad (13)$$

is the noise response.

ii.a White noise Consider first white noise, defined through $\langle f_0(t_1) f_0(t_2) \rangle = f_0^2 T \delta(t_1 - t_2)$, where we have introduced a unit of time T for dimensional grounds, in addition to the force amplitudes f_0 . One then immediately gets

$$\sigma^2(q_0(t)) = f_0^2 T \int_0^t \frac{\sin^2 \sqrt{\lambda_0}(t-t')}{\lambda_0} dt' \leq \frac{f_0^2 T t}{\lambda_0}. \quad (14)$$

In fact, for large times, $t \gg 1/\sqrt{\lambda_0}$, one has $\sigma^2(q_0(t)) \simeq \frac{f_0^2 T t}{2\lambda_0}$. All the dependence on N of the sensitivity arises again from the derivative of $\langle q_0(t) \rangle$ and thus λ_0 with respect to ξ^2 . Inserting everything in eq.(7), we are led to

$$\delta\xi_{min}^2 \leq \frac{2f_0 \sqrt{T/t}}{\sqrt{M} N |q_0(0) \sin(\sqrt{\lambda_0}t)|}, \quad (15)$$

where for large times still a factor $1/\sqrt{2}$ can be gained on the rhs. Again, we see that the result scales as $1/N$. The time dependence is different from the previous case (and the typical quantum situation at the HL): for large times the smallest resolvable $\delta\xi^2$ decays only as $1/\sqrt{t}$, just as in the standard quantum limit.

ii.b Colored noise. The above considerations are easily generalized to colored noise. In fact, unless the noise f_0 on oscillator 0 depends already at order ξ^0 on N (which appears to be a highly artificial situation, since without interaction the central oscillator should not “know” about the number of additional oscillators), $\sigma^2(n(t))$ is independent of N , and the same scaling analysis concerning

N therefore applies and always leads to a $1/N$ scaling of the sensitivity. Only the time dependence will differ. As a more general example, consider stationary colored noise with a correlation function $\langle f_0(t_1)f_0(t_2) \rangle = f_0^2 C(t_1 - t_2)$ where we take $C(0) = 1$, and $C(-t) = C(t)$. One then easily finds the upper bound

$$\sigma^2(q_0(t)) \leq \frac{2f_0^2}{\lambda_0} \int_0^t dt_- |C(t_-)|(t - t_-). \quad (16)$$

If the correlation function vanishes for $t > t_c$, one has $\sigma^2(q_0(t)) \leq \frac{2f_0^2}{\lambda_0} b(t)$ with

$$b(t) = \begin{cases} tt_c - \frac{1}{2}t_c^2 & t_c < t \\ \frac{1}{2}t^2 & t_c \geq t, \end{cases} \quad (17)$$

where we have used $|C(t)| \leq |C(0)|$. Correspondingly, we have for the sensitivity

$$\delta\xi_{\min}^2 \leq \frac{2\sqrt{2}f_0}{\sqrt{M}} \frac{\sqrt{b(t)}}{Nt|q_0(0)\sin(\sqrt{\lambda_0}t)|}, \quad (18)$$

which again scales as $1/N$.

A possible application of this “coherent averaging” technique might be a novel way of measuring the gravitational constant G , which is one of the least well-determined natural constants with a relative uncertainty of order 10^{-4} and the 1986 CODATA recommended value based on conflicting experimental results [41]. One of the reasons for this dire situation is the extremely weak strength of the gravitational interaction. This, and the impossibility to shield the gravitational field from other disturbing bodies, render the determination of the absolute value of G very difficult, in spite of continued strong interest, driven in part by attempts to detect a variation of G as function of distance, time, or other physical quantities. Since Cavendish’s pioneering work in 1798, essentially all lab-experiments attempting to measure G were based on a beam balance or a torsion pendulum, and measured either a static response (some by counterbalancing deflections of the small test masses), or a dynamic one (allowing frequency-specific analysis synchronized with a periodic excitation, see e.g. the recent attempt to measure deviations of the $1/r^2$ behavior below the dark energy length scale of about $85\mu\text{m}$ [42]). Our “coherent averaging” method suggests a new, massively parallel way of attempting to measure G more precisely: Couple $N \gg 1$ torsion balances gravitationally to one central one (consisting of N further beams with masses fixed at the ends, and all beams attached rigidly to a common axis). Then measure the shift in frequency of that central oscillator as function of N and the positions of the test masses. This will enable a reduction of the uncertainty in the measured value of G with a scaling $1/N$ and should become competitive with traditional measurements for large N .

In summary, the above analysis shows that even in the classical realm there are situations where it is possible to beat the venerated procedure of averaging N measurement results of a physical quantity that leads to a $1/\sqrt{N}$ scaling of the sensitivity. It can be improved upon by coupling the N samples “coherently” to a central oscillator that will pick up a collective phase proportional to N and the parameter to be measured, and yield in the end a $1/N$ scaling with the number of samples available. Applications might be found in the measurement of very weak interactions, such as the gravitational interaction between lab-scale test masses, suggesting a new, massively parallel way of determining the gravitational constant.

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